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Groups, Geometry, and Dynamics

Free subgroups of 3-manifold groups

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Abstract. We show that any closed hyperbolic 3-manifold M has a co-final tower of covers $M_i \to M$ of degrees n_i such that any subgroup of $\pi_1(M_i)$ generated by k_i elements is free, where $k_i \geq n_i^C$ and $C = C(M) > 0$. Together with this result we prove that $\log k_i \geq C_1 \operatorname{sys}_1(M_i)$, where $\operatorname{sys}_1(M_i)$ denotes the systole of M_i, thus providing a large set of new examples for a conjecture of Gromov. In the second theorem $C_1 > 0$ is an absolute constant. We also consider a generalization of these results to non-compact finite volume hyperbolic 3-manifolds.

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1. Introduction

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a cocompact Kleinian group and $M = \mathbb{H}^3/\Gamma$ be the associated quotient space. It is a closed orientable hyperbolic 3-orbifold, it is a manifold if Γ is torsion-free. We will call a group Γ *k-free* if any subgroup of Γ generated by k elements is free. We denote the maximal k for which Γ is k-free by $\mathcal{N}_{fr}(\Gamma)$ and we call it the *free rank* of Γ. For example, if S_g is a closed Riemann surface of genus g, then its fundamental group satisfies $\mathcal{N}_{fr}(\pi_1(S_g)) = 2g-1$. In this note we prove that for any Kleinian group as above there exists an exhaustive filtration of normal subgroups Γ_i of Γ such that $\mathcal{N}_{fr}(\Gamma_i) \geq [\Gamma : \Gamma_i]^C$, where $C = C(\Gamma) > 0$ is a constant. In geometric terms the result can be stated as follows.

Theorem 1. *Let M be a closed hyperbolic 3-orbifold. Then there exists a co-final tower of regular finite-sheeted covers $M_i \to M$ such that*

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq \operatorname{vol}(M_i)^C,$$

where $C = C(M)$ is a positive constant which depends only on M.

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The proof of the theorem is based on the previous results of Baumslag, Shalen and Wagreich [4, 18], Belolipetsky [5], and Calegari and Emerton [6]. Let us emphaisize that although some of the results use arithmetic techniques, our theorem applies to *all* closed hyperbolic 3-orbifolds. A result of similar flavor but for another property of 3-manifold groups was obtained by Long, Lubotzky, and Reid in [15]. Indeed, in some parts our construction comes close to their argument.

Together with Theorem 1 we obtain the following theorem of independent interest:

Theorem 2. *Any closed hyperbolic* 3*-orbifold admits a sequence of regular manifold covers* $M_i \to M$ *such that*

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq (1+\varepsilon)^{\mathrm{sys}_1(M_i)},$$

where $\varepsilon > 0$ *is an absolute constant and* $\mathrm{sys}_1(M_i)$ *is the length of a shortest closed geodesic in* M_i.

This type of bound was stated by Gromov [8, Section 5.3.A] for hyperbolic groups in general, but later turned into a conjecture (see [9, Section 2.4]). We refer to the introduction of [5] for a related discussion and some other references. In [9], Gromov particularly mentioned that the conjecture is open even for hyperbolic 3-manifold groups. The first set of examples of hyperbolic 3-manifolds for which the conjecture is true was presented in [5]. These examples were all arithmetic. Our theorem significantly enlarges this set.

We review the construction of covers $M_i \to M$ and prove a lower bound for their systoles in Section 2. Theorems 1 and 2 are proved in Section 3. In Section 4 we consider a generalization of the results to non-compact finite volume 3-manifolds. Their groups always contain a copy of $\mathbb{Z} \times \mathbb{Z}$, so have $\mathcal{N}_{fr} = 1$, however, we can modify the definition of the free rank so that it becomes non-trivial for the non-compact manifolds: we define $\mathcal{N}'_{fr}(\Gamma)$ to be the maximal k for which the group Γ is k-semifree, where Γ is called *k-semifree* if any subgroup generated by k elements is a free product of free abelian groups. With this definition at hand we can extend Gromov's conjecture to the groups of finite volume non-compact manifolds. In Section 4 we prove:

Theorem 3. *Any finite volume hyperbolic* 3*-orbifold admits a sequence of regular manifold covers* $M_i \to M$ *such that*

$$\mathcal{N}'_{fr}(\pi_1(M_i)) \geq (1+\varepsilon)^{\mathrm{sys}_1(M_i)},$$

where $\varepsilon > 0$ *is an absolute constant.*

To conclude the introduction we would like to point out one important detail. While in Theorems 2 and 3 we have an absolute constant $\varepsilon > 0$, the constant in

Theorem 1 depends on the base manifold. In [5] it was shown that in arithmetic case $C(M)$ is also bounded below by a universal positive constant. Existence of a bound of this type in general remains an open problem.

Question 1. *Do there exist an absolute constant $C_0 > 0$ such that for any M in Theorem 1 we have $C(M) \geq C_0$.*

2. Preliminaries

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a *lattice*, i.e. a finite covolume discrete subgroup. By Mostow–Prasad rigidity, Γ admits a discrete faithful representation into $\mathrm{SL}_2(\mathbb{C})$ with the entries in some (minimal) number field E. Since Γ is finitely generated, there is a finite set of primes S in E such that $\Gamma < \mathrm{SL}_2(\mathcal{O}_{E,S})$, where $\mathcal{O}_{E,S}$ denotes the ring of S-integers in E.

Following Calegari and Emerton [6], we can consider an exhaustive filtration of normal subgroups Γ_i of Γ which gives rise to a co-final tower of hyperbolic 3-manifolds covering $\mathbb{H}^3/\Gamma$. The subgroups Γ_i are defined as follows. From the description of Γ given above it follows that it is residually finite and for all but finitely many primes $\mathfrak{p} \in \mathcal{O}_E$ there is an injective map $\phi_{\mathfrak{p}}\colon \Gamma \to \mathrm{SL}_2(\widehat{\mathcal{O}}_{E,\mathfrak{p}})$ (where $\widehat{\mathcal{O}}_{E,\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of the ring of integers of E). Let p be a rational prime such that for any prime $\mathfrak{p}$ in $\mathcal{O}_E$ which divides p, the correspondent map $\phi_{\mathfrak{p}}$ is injective (this holds for almost all primes p). We can write $p\mathcal{O}_E = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_{\mathrm{m}}{}^{e_m}$.

For any $j = 1, \dots, m$, the ring $\widehat{\mathcal{O}}_{E,\mathfrak{p}_j}$ contains $\mathbb{Z}_p$ as a subring and is a $\mathbb{Z}_p$-module of dimension $d_j = e_j f_j$, where f_j is the degree of the extension of residual fields $[\mathcal{O}_E/\mathfrak{p}_j : \mathbb{Z}/p\mathbb{Z}]$. If we fix j and a basis $b_1^j, \dots, b_{d_j}^j \in \widehat{\mathcal{O}}_{E,\mathfrak{p}_j}$ as $\mathbb{Z}_p$-module, we have a natural ring homomorphism $\psi_j\colon \widehat{\mathcal{O}}_{E,\mathfrak{p}_j} \to M_{d_j \times d_j}(\mathbb{Z}_p)$ given by $\psi_j(x) = (x_{rs})$ if $x b_s^j = \sum_{r=1}^{d_j} x_{rs} b_r^j$.

Let $\psi\colon \prod_{j=1}^m \mathrm{SL}_2(\widehat{\mathcal{O}}_{E,\mathfrak{p}_j}) \to \mathrm{GL}_N(\mathbb{Z}_p)$ be given diagonally by the blocks $\psi_1, \dots, \psi_m$, where $N = 2\sum_j d_j$. Let $\phi = \psi \circ \prod_{j=1}^m \phi_{\mathfrak{p}_j}\colon \mathrm{SL}_2(\mathcal{O}_{E,S}) \to \mathrm{GL}_N(\mathbb{Z}_p)$. The Zariski closure of the image of ϕ is a group $G < \mathrm{GL}_N(\mathbb{Z}_p)$ of dimension $d \geq 6$ (cf. [6, Example 5.7]). It is a p-adic analytic group which admits a normal exhaustive filtration

$$G_i = G \cap \ker\big(\mathrm{GL}_N(\mathbb{Z}_p) \longrightarrow \mathrm{GL}_N(\mathbb{Z}_p/p^i\mathbb{Z}_p)\big).$$

This filtration gives rise to a filtration of Γ via the normal subgroups $\Gamma_i = \phi^{-1}(G_i)$. The filtration (Γ_i) is exhaustive because ϕ is injective.

Associated to each of the subgroups Γ_i of Γ is a finite-sheeted cover M_i of $M = \mathbb{H}^3/\Gamma$, and by the construction the sequence (M_i) is a co-final tower of covers of M. By Minkowski's lemma, almost all groups G_i are torsion-free, hence associated M_i are smooth hyperbolic 3-manifolds. Therefore, when it is needed we can assume that M is a manifold itself.

We will require a lower bound for the systole of M_i. Such a bound is essentially provided by Proposition 10 of [10], which can be seen as a generalization of a result of Margulis [16] (see also [15]). The main difference is that we do not restrict to arithmetic manifolds. The main technical difference is that while in [op. cit.] the authors consider matrices with real entries we do it for p-adic numbers, which requires replacing norm of a matrix by the height of a matrix. This technical part is more intricate, however, as it is shown below, it does not affect the main argument.

Lemma 2.1. *Suppose M is a compact manifold. Then there is a constant $c_1 = c_1(M) > 0$ such that $\mathrm{sys}_1(M_i) \geq c_1 \log n_i$, where $n_i = [\Gamma : \Gamma_i]$.*

Proof. Since M is compact, we can apply the Milnor–Schwarz lemma. Therefore, if we fix a point $o \in \mathbb{H}^3$, then Γ has a finite symmetric set of generators X such that the map $(\Gamma, X) \to \mathbb{H}^3$ given by $\gamma \mapsto \gamma(o)$ is a (C_1, C_2) quasi-isometry. This means that for any pair $\gamma_1, \gamma_2 \in \Gamma$ we have

$$C_1 d_X(\gamma_1, \gamma_2) - C_2 \leq d(\gamma_1(o), \gamma_2(o)) \leq \frac{1}{C_1} d_X(\gamma_1, \gamma_2) + C_2,$$

where $d(\cdot, \cdot)$ denotes the distance function in $\mathbb{H}^3$, $d_X(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|_X$ and $|\gamma|_X$ is the minimal length of a word in X which represents γ. For any $i \geq 1$, we define $\mathrm{sys}(\Gamma_i, X) = \min\{d_X(1, \gamma) \mid \gamma \in \Gamma_i \backslash \{1\}\}$.

Claim 1. Let $\delta_M > 0$ be the diameter of M. For any $i \geq 1$, we have

$$\mathrm{sys}_1(M_i) \geq C_1 \mathrm{sys}(\Gamma_i, X) - C_2 - 2\delta_M.$$

To prove the claim, consider the Dirichlet fundamental domain $D(o)$ of Γ in $\mathbb{H}^3$ centered in o. It is easy to see that any point $x \in D(o)$ satisfies $d(x, o) \leq \delta_M$. Now let $\alpha_i \subset M_i$ be a closed geodesic realizing the systole of M_i. As $M_i \to M$ is a local isometry, the image of α_i in M has the same length (counted with multiplicity). Denote the image by α_i again. We can suppose that $x_i \in D(o)$ is a lift of $\alpha_i(0)$. Thus, there exists a unique nontrivial $\gamma_i \in \Gamma_i$ such that $\mathrm{sys}_1(M_i) = d(x_i, \gamma_i(x_i))$. Note that $d(x_i, o) = d(\gamma_i(x_i), \gamma_i(o)) \leq \delta_M$, therefore, by the triangle inequality we have

$$\begin{aligned} \mathrm{sys}_1(M_i) &\geq d(o, \gamma_i(o)) - 2\delta_M \\ &\geq C_1 d_X(1, \gamma_i) - C_2 - 2\delta_M \\ &\geq C_1 \mathrm{sys}(\Gamma_i, X) - C_2 - 2\delta_M. \end{aligned}$$

Now our problem is reduced to proving that $\mathrm{sys}(\Gamma_i, X)$ grows logarithmically as a function of $[\Gamma : \Gamma_i]$. In order to do so we use arithmetic of the field E in an essential way.

Let $S(E)$ be the set of all places of E, S_∞ be the set of archimedean places, and S_p be the set of places corresponding to the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$, which appear in the definition of M_i. For any $x \in E$, we define the *height* of x by $H(x) = \prod_{v \in S(E)} \max\{1, |x|_v\}$. Recall that for any $x, y \in E$ and an archimedean place v, we have $|x+y|_v \leq 4\max\{|x|_v, |y|_v\}$, and for any non-archimedean place u, we have $|x+y|_u \leq \max\{|x|_v, |y|_u\}$. Therefore, the height function satisfies $H(x+y) \leq 4^{\#S_\infty} H(x)H(y)$.

We can generalize the definition of height for matrices with entries in E. Thus, for any $M = (m_{ij}) \in \mathrm{SL}_2(E)$, we define $H(M) = \prod_{v \in S(E)} \max\{1, |m_{ij}|_v\}$. We note that $H(M) \geq \max\{H(m_{ij})\}$.

Claim 2. For any $M, N \in \mathrm{SL}_2(E)$, we have $H(MN) \leq 4^{\#S_\infty} H(M)H(N)$.

Indeed, any entry x of MN can be written as $x = au + bt$ with a, b entries of M and u, t entries of N. Therefore, for any $v \in S_\infty$,

$$\begin{aligned}\max\{1, |x|_v\} &\leq 4\max\{1, |a|_v, |b|_v\}\max\{1, |u|_v, |t|_v\} \\ &\leq 4\max\{1, |m_{ij}|_v\}\max\{1, |n_{ij}|_v\}.\end{aligned}$$

For non-archimedean places we have the same inequality without the factor 4. Now if $MN = (x_{ij})$, then these inequalities show that

$$H(MN) = \prod_{v \in S(E)} \max\{1, |x_{ij}|_v\} \leq 4^{\#S_\infty} H(M)H(N).$$

Next we want to estimate from below the height of γ for any nontrivial $\gamma \in \Gamma_i$.

Claim 3. There exists a constant $C_3 > 0$ such that for any $\gamma \in \Gamma_i \backslash \{1\}$ we have $H(\gamma) \geq C_3 p^{ni}$, where $n = [E : \mathbb{Q}]$.

Indeed, let $\gamma = \gamma_{r_1} \cdots \gamma_{r_{w(\gamma)}} \in \Gamma_i$ be a nontrivial element with $\gamma_{r_j} \in X$ and $w(\gamma) = d_X(1, \gamma)$. We now recall the definition of the group Γ_i. If we write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then for any $l = 1, \ldots, m$ we have

$$\begin{pmatrix} \psi_l(a) & \psi_l(b) \\ \psi_l(c) & \psi_l(d) \end{pmatrix} \equiv \begin{pmatrix} I_{d_l} & 0 \\ 0 & I_{d_l} \end{pmatrix} \mod (p^i \mathbb{Z}_p).$$

By the definition of ψ_l we have that $(a-1)b_j^l, bb_j^l, cb_j^l, (d-1)b_j^l \in p^i \widehat{\mathcal{O}}_{E,\mathfrak{p}_l}$ for any $1 \leq j \leq d_l$. Taking $C^* = \min_{l,j}\{|b_j^l|_{\mathfrak{p}_l}\} > 0$, we obtain

$$C^* \max\{|a-1|_{\mathfrak{p}_l}, |b|_{\mathfrak{p}_l}, |c|_{\mathfrak{p}_l}, |d-1|_{\mathfrak{p}_l}\} \leq \mathrm{Norm}(\mathfrak{p}_l)^{-ie_l},$$

for any $l = 1, \ldots, m$. This is because $|p|_{\mathfrak{p}_l} = \mathrm{Norm}(\mathfrak{p}_l)^{-e_l}$ by definition, where for an ideal $I \subset \mathcal{O}_E$ the *norm* of I is equal to $\mathrm{Norm}(I) = \#(\mathcal{O}_E/I)$.

Recall that the Product Formula says that for any nonzero $x \in E$ we have $\prod_v |x|_v = 1$. Since γ is nontrivial, at least one of the numbers $\{a-1, b, c, d-1\}$ is not zero. Therefore, if we apply the Product Formula for any nonzero element in this set, we obtain

$$\max\{H(a-1), H(b), H(c), H(d-1)\} \geq \prod_{l=1}^{m} C^* \mathrm{Norm}(\mathfrak{p}_l)^{ie_i} = (C^*)^m p^{ni}.$$

Moreover, by the estimate of the height of a sum we have

$$\max\{H(a-1), H(b), H(c), H(d-1)\} \leq 4^{\#S_\infty} \max\{H(a), H(b), H(c), H(d)\},$$

therefore,

$$H(\gamma) \geq \max\{H(a), H(b), H(c), H(d)\} \geq \frac{(C^*)^m p^{ni}}{4^{\#S_\infty}} = C_3 p^{ni}.$$

This proves Claim 3.

We can now finish the proof of the lemma. If we take

$$C_4 = 4^{\#S_\infty} \max\{H(M) \mid M \in X\},$$

we have

$$C_3 p^{ni} \leq H(\gamma) \leq (4^{\#S_\infty})^{d_X(1,\gamma)-1} (\max\{H(M) \mid M \in X\})^{d_X(1,\gamma)} \leq C_4^{d_X(1,\gamma)}.$$

This estimate holds for any nontrivial $\gamma \in \Gamma_i$, hence $C_3 p^{ni} \leq C_4^{\mathrm{sys}(\Gamma_i, X)}$ for any i. On the other hand, there exists a constant $C_5 > 0$ such that $[\Gamma : \Gamma_i] \leq C_5 p^{i \dim(G)}$. These inequalities together imply that

$$\mathrm{sys}(\Gamma_i, X) \geq \frac{n}{\dim(G) \log(C_4)} \log([\Gamma : \Gamma_i]) + \frac{\log(C_3 C_5^{\frac{-n}{\dim G}})}{\log(C_4)}.$$

Since $[\Gamma : \Gamma_i] \to \infty$ and $\mathrm{sys}_1(M_i)$ is bounded below by a positive constant, we conclude that there exists a constant $c_1 = c_1(o, \delta_M, p, \psi_1, \ldots, \psi_m) = c_1(M) > 0$ such that $\mathrm{sys}_1(M_i) \geq c_1 \log([\Gamma : \Gamma_i])$ for any $i \geq 1$. □

Note that the constant c_1 depends on M (cf. Question 1). If M is arithmetic, then by [13] we can take $c_1 = \frac{2}{3} - \epsilon$ for a small $\epsilon > 0$ assuming n_i is sufficiently large. In general case the argument of [13] does not apply, while the proof of Lemma 2.1 does not provide a sufficient level of control over the constants.

3. Proofs of Theorems 1 and 2

Following [5], we define the *systolic genus* of a manifold M by

$$\mathrm{sysg}(M) = \min\{g \mid \text{the fundamental group } \pi_1(M) \text{ contains } \pi_1(S_g)\},$$

where S_g denotes a closed Riemann surface of genus $g > 0$.

Let M be a closed hyperbolic 3-manifold with sufficiently large systole $\mathrm{sys}_1(M)$. By [5, Theorem 2.1], we have

$$\log \mathrm{sysg}(M) \geq c_2 \cdot \mathrm{sys}_1(M), \tag{1}$$

where $c_2 > 0$ is an absolute constant (for any $\delta > 0$, assuming $\mathrm{sys}_1(M)$ is sufficiently large, we can take $c_2 = \frac{1}{2} - \delta$).

The second ingredient of the proof is a theorem of Calegary and Emerton [6], which implies that for the sequences of covers defined in Section 2 we have

$$\dim \mathrm{H}_1(M_i, \mathbb{F}_p) \geq \lambda \cdot p^{(d-1)i} + O(p^{(d-2)i}) \tag{2}$$

for some rational constant $\lambda \neq 0$. Recall that we have dimension $d = \dim(G) \geq 6$ and the degree of the covers $M_i \to M$ grows like p^{di}. Hence we can rewrite (2) in the form

$$\dim \mathrm{H}_1(M_i, \mathbb{F}_p) \geq c_3 \mathrm{vol}(M_i)^{5/6}, \tag{3}$$

where $c_3 > 0$ is a constant depending on M and we assume that $\mathrm{vol}(M_i)$ is sufficiently large.

We note that in contrast with the previous related work, the theorem of [6] applies to non-arithmetic manifolds as well as to the arithmetic ones.

Now recall a result of Baumslag–Shalen [4, Appendix]. They show that if $\mathrm{sysg}(M) \geq k$ and $\dim \mathrm{H}_1(M, \mathbb{Q}) \geq k + 1$, then $\pi_1(M)$ is k-free. In a subsequent paper [18], Shalen and Wagreich proved that the same conclusion holds if $\mathrm{sysg}(M) \geq k$ and $\dim \mathrm{H}_1(M, \mathbb{F}_p) \geq k + 2$ [loc. cit., Proposition 1.8].

We now bring all the ingredients together. Given a closed hyperbolic 3-orbifold M, for the sequence (M_i) of its manifold covers defined in Section 2 we have

$$\begin{aligned} \mathrm{sysg}(M_i) &\geq e^{c_2 \cdot \mathrm{sys}_1(M_i)} && \text{(by (1))} \\ &\geq \mathrm{vol}(M_i)^c && \text{(by Lemma 2.1);} \end{aligned}$$

and

$$\dim \mathrm{H}_1(M_i, \mathbb{F}_p) \geq c_3 \cdot \mathrm{vol}(M_i)^{5/6} \quad \text{(by (3)).}$$

Hence by the theorem from [18] cited above we obtain

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq \mathrm{vol}(M_i)^C,$$

where $C = C(M) > 0$ and we assume that $\mathrm{vol}(M_i)$ is sufficiently large. This proves Theorem 1.

For the second theorem recall that the systole of a hyperbolic 3-manifold is bounded above by the logarithm of its volume. Indeed, a manifold M with a systole $\mathrm{sys}_1(M)$ contains a ball of radius $r = \mathrm{sys}_1(M)/2$. The volume of a ball in $\mathbb{H}^3$ is given by $\mathrm{vol}(B(r)) = \pi(\sinh(2r) - 2r)$, hence we get

$$\mathrm{vol}(M) \geq \pi(\sinh(\mathrm{sys}_1(M)) - \mathrm{sys}_1(M)) \sim \frac{\pi}{2} e^{\mathrm{sys}_1(M)};$$
$$\mathrm{vol}(M) \geq e^{c \cdot \mathrm{sys}_1(M)}, \quad \text{as } \mathrm{sys}_1(M) \to \infty.$$

By Lemma 2.1, the systole of the covers $M_i \to M$ grows as $i \to \infty$. Therefore, we can bound both $\mathrm{sysg}(M_i)$ and $\dim \mathrm{H}_1(M_i, \mathbb{F}_p)$ below by an exponential function of $\mathrm{sys}_1(M_i)$ with an absolute constant in exponent. Theorem 2 now follows immediately from the theorem of [18]. □

Remark 3.1. It follows from the proof that for any $\delta > 0$, assuming $\mathrm{sys}_1(M_i)$ is large enough, we can take ε in Theorem 2 equal to $e^{\frac{1}{2}-\delta} - 1$. The same bound applies for the constant in Theorem 3, which we prove in the next section.

4. Generalization to finite volume hyperbolic 3-manifolds

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a finite covolume Kleinian group. The quotient $M = \mathbb{H}^3/\Gamma$ is a finite volume orientable hyperbolic 3-orbifold, which can be either closed or non-compact with a finite number of cusps. The group Γ is a relatively hyperbolic group with respect to the cusp subgroups. In this section we discuss a generalization of Gromov's conjecture and our results to this class of groups.

We call Γ a *k-semifree group* if any subgroup of Γ generated by k elements is a free product of free abelian groups. The maximal k for which Γ is k-semifree is denoted by $\mathcal{N}'_{fr}(\Gamma)$. With this definition, we can generalize Gromov's conjecture to relatively hyperbolic groups. Although the injectivity radius of manifolds with cusps vanish, their systole is still bounded away from zero. Therefore, a natural generalization of Gromov's conjecture would be that $\mathcal{N}'_{fr}(\Gamma)$ is bounded below by an exponential function of the systole of the associated quotient space M. Theorem 3, which we prove in this section, can be considered as an evidence for this conjecture.

We need to modify the definition of the *systolic genus* of a manifold M in the following way:

$$\mathrm{sysg}(M) = \min\{g > 1 \mid \text{the fundamental group } \pi_1(M) \text{ contains } \pi_1(S_g)\},$$

where S_g denotes a closed Riemann surface of genus g. We excluded the genus $g = 1$ in order to adapt the definition to the non-compact finite volume 3-manifolds which otherwise would all have $\mathrm{sysg} = 1$.

Let M be a finite volume hyperbolic 3-manifold with sufficiently large systole $\mathrm{sys}_1(M)$. By [5, Theorem 2.1], if M is closed, we have

$$\log \mathrm{sysg}(M) \geq c_2 \cdot \mathrm{sys}_1(M), \tag{4}$$

where $c_2 > 0$ is an absolute constant. We now discuss a generalization of this result to non-compact finite volume 3-manifolds. The first step in the proof of the theorem in [5] is an application of the theorem of Schoen–Yau and Sacks–Uhlenbeck, which allows to homotop a π_1-injective map of a surface of genus $g > 1$ into M to a minimal immersion. This result was recently generalized to the finite volume hyperbolic 3-manifolds in the work of Collin–Hauswirth–Mazet–Rosenberg [7] and Huang–Wang [11] (see in particular [11, Theorem 1.1]). So let S_g be a closed immersed least area minimal surface in M. In order to establish (4) for M we can suppose that S_g is embedded. Indeed, since $\pi_1(M)$ is *LERF* [2, Corollary 9.4] there exists a finite covering $\tilde{M}$ of M such that S_g is embedded and π_1-injective in $\tilde{M}$. Moreover, $g \geq \mathrm{sysg}(\tilde{M})$ and $\mathrm{sys}_1(\tilde{M}) \geq \mathrm{sys}_1(M)$. If S_g has no accidental parabolic curves, then the systole of S_g with respect to the induced metric satisfies $\mathrm{sys}_1(S_g) \geq \mathrm{sys}_1(M)$ and the rest of the proof in [5] applies without any changes.

In the presence of accidental parabolics, we can apply the following lemma.

Lemma 4.1 (Compression Lemma). *Let M be a non-compact hyperbolic 3-manifold of finite volume. Suppose that there exists a π_1-injective embedded closed surface $S_g \subset M$, for some genus $g \geq 2$, such that S_g has an accidental parabolic simple curve α. Then there exist disjoint tori $T_1, \dots, T_n \subset M$, one for each cusp $\mathcal{C}(T_i)$ of M, such that the compact 3-manifold $M' = M \setminus \cup_{i=1}^{n} \mathcal{C}(T_i)$ has a properly incompressible and boundary-incompressible surface $S_{g',p}$ with $g' \geq \frac{g}{2}$ and $1 \leq p \leq 2$.*

Proof. Suppose that α is associated to a parabolic isometry corresponding to a cusp $\mathcal{C} = T_0 \times [0, \infty)$ of M, where T_0 is a maximal torus. Since S_g is compact we can consider a torus $T = T_0 \times \{t_0\} \subset \mathcal{C}$ for some $t_0 > 0$ sufficiently large such that $S_g \subset M \setminus T_0 \times [t_0, \infty)$. We denote by $\beta \subset T$ the corresponding simple curve homotopic to α.

We first show that there exists an embedding $f : S_g \to M$ homotopic to the embedding $\iota : S_g \to M$ such that f is transversal to some torus $T_1 \subset \mathcal{C}$ and $f(S_g) \cap T_1 \times [0, \infty) \subset \mathcal{C}$ is an annulus with boundary curves $f(\alpha_0)$, $f(\alpha_1)$, where α_0, α_1 are the boundary curves of a collar neighborhood of α in S_g.

As an application of the Jaco–Shalen Annulus Theorem [12, Theorem VIII.13], there exists an embedding $H_0 : \mathbb{S}^1 \times [0, 1] \to M$ with $H(\theta, 0) = \alpha(\theta)$ and $H(\theta, 1) = \beta(\theta)$ (see [17, Lemma 2.1]). We can suppose that H_0 is transversal to S_g and T and is such that if we denote by $\mathcal{A}$ the image $H_0(\mathbb{S}^1 \times [0, 1])$, then $\mathcal{A} \cap S_g = \alpha$ and $\mathcal{A} \cap M \setminus T \times [0, \infty) = \beta$.

Let $\mathcal{D}$ be a collar neighborhood of α in S_g contained in a tubular neighborhood $\pi\colon E \subset M \to \mathcal{A}$ such that $\mathcal{D} \cap \mathcal{A} = \alpha$. Since $\pi\colon E \to \mathcal{A}$ is trivial, we can deform $\mathcal{D}$ into E preserving the boundary and moving α along $\mathcal{A}$. We get a new annulus $\mathcal{D}' \subset M$ with $\partial\mathcal{D}' = \alpha_0 \cup \alpha_1$ and $\mathcal{D}' \cap T = \beta$.

Let ψ be the diffeomorphism between $\mathcal{D}$ and $\mathcal{D}'$ given by the deformation. We can suppose that ψ is the identity in a small neighborhood of the boundary. We now define the map $f\colon S_g \to M$ by $f(x) = x$ if $x \notin \mathcal{D}$ and $f(y) = \psi(y)$ if $y \in \mathcal{D}$. It is a smooth embedding homotopic to the inclusion.

By transversality, for some $0 < t_1 < t_0$ we have a torus $T_1 = T_0 \times \{t_1\}$ and a subannulus $\widehat{\mathcal{D}} \subset \mathcal{D}$ such that f is transversal to T_1 and

$$f(S_g) \cap M \setminus T_1 \times [0,\infty) = f(S_g \setminus \operatorname{int}(\widehat{\mathcal{D}})) \quad \text{and} \quad f(\partial(S_g \setminus \operatorname{int}(\widehat{\mathcal{D}}))) = f(\partial\widehat{\mathcal{D}}) \subset T_1.$$

This shows that embedding f has the desired properties.

Now, for the torus T_1 constructed above, there exist disjoint tori $T_2, \dots, T_n$ in the cusps of M such that the corresponding cusps $\mathcal{C}(T_j) \cap \mathcal{C}(T_1) = \emptyset$ for all $j = 2, \dots, n$ and $f(S_g \setminus \operatorname{int}(\widehat{\mathcal{D}})) \subset M' = M \setminus \cup_{i=1}^{n} \mathcal{C}(T_i)$, and we have that $f(S_g \setminus \operatorname{int}(\widehat{\mathcal{D}})) \subset M'$ is a proper submanifold of M'.

Note that $f(S_g \setminus \operatorname{int}(\widehat{\mathcal{D}}))$ is connected with two boundary curves if α does not separate and has two components with a boundary curve if α separates it. In the latter case we consider the component with the maximal genus. Hence in both cases we have a surface $S_{g',p}$ with $g' \geq \frac{g}{2}$ and $1 \leq p \leq 2$ and a proper embedding $f\colon S_{g',p} \to M'$.

Recall that a properly embedded surface F in a compact 3-manifold N with boundary is called *boundary-compressible* if either F is a disk and F is parallel to a disk in ∂N, or F is not a disk and there exists a disk $D \subset N$ such that $D \cap F = c$ is an arc in ∂D, $D \cap \partial N = c'$ is an arc in ∂D, with $c \cap c' = \partial c = \partial c'$ and $c \cup c' = \partial D$, and either c does not separate F or c separates F into two components and the closure of neither is a disk. Otherwise, F is *boundary-incompressible* (see [12, Chapter III]).

Since $S_g \subset M$ is π_1-injective, it follows from the definition and our construction that $S_{g',p} \subset M'$ is incompressible and boundary-incompressible. □

We now apply to $S_{g',p}$ a result of Adams and Reid [1, Theorem 5.2]. Since $\operatorname{sys}_1(M) = \operatorname{sys}_1(M')$, it immediately implies inequality (4).

The theorem of Calegary–Emerton applies to non-cocompact groups as well as to the cocompact ones.

We finally recall a result of Anderson–Canary–Culler–Shalen [3]. They show that if $\operatorname{sysg}(M) \geq k$ and $\dim \mathrm{H}_1(M, \mathbb{F}_p) \geq k + 2$ for some prime p, then $\pi_1(M)$ is k-semifree [loc. cit., Corollary 7.4]. This theorem generalizes the previous results in [4, 18] to non-compact hyperbolic 3-manifolds. Its proof also makes an essential use of topology of 3-manifolds.

Similar to Section 3, we bring together all the ingredients considered above.

Given a finite volume hyperbolic 3-orbifold M, for the sequence (M_i) of its manifold covers defined in Section 2 we have

$$\operatorname{sysg}(M_i) \geq e^{c_2 \cdot \operatorname{sys}_1(M_i)} \quad \text{(by (4))},$$

and

$$\dim \mathrm{H}_1(M_i, \mathbb{F}_p) \geq c_3 \cdot \operatorname{vol}(M_i)^{5/6} \quad \text{(by Calegary–Emerton)}.$$

The fact that a manifold M with systole $\operatorname{sys}_1(M)$ contains a ball of radius $r = \operatorname{sys}_1(M)/2$ is not necessarily true for non-compact finite volume hyperbolic 3-manifolds but it is still possible to bound the volume by an exponential function of the systole. By Lakeland–Leininger [14, Theorem 1.3], we have

$$\operatorname{vol}(M) \geq e^{c \cdot \operatorname{sys}_1(M)}, \quad \text{as } \operatorname{sys}_1(M) \to \infty$$

(with $c = \frac{3}{4} - \delta$ for any $\delta > 0$, assuming $\operatorname{sys}_1(M)$ is sufficiently large).

Although we do not have a generalization of Lemma 2.1, we do know that $\operatorname{sys}_1(M_i) \to \infty$ with i because the sequence of covers $M_i \to M$ is co-final. Therefore, we can bound both $\operatorname{sysg}(M_i)$ and $\dim \mathrm{H}_1(M_i, \mathbb{F}_p)$ below by an exponential function of $\operatorname{sys}_1(M_i)$ with an absolute constant in exponent and Theorem 3 now follows from the theorem of [3]. □

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